Spin-down in the absence of reflected waves

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Overview

Project Aims

- To reproduce the results detailed by Li, Patterson, Zhang and Kerswell in their paper: "*Spin-up and spin-down in a half cone: A pathological situation or not?*" [1] using simpler techniques.
- To also look for anything markedly different between both sets of findings and discuss.

What did they do?

- Identify how initial vorticity generated by spin-up/down in a half cone propagates and over what time scale it decays.
- Argued that no discrete set of oscillatory modes exist in a half cone because the vertex "precludes their existence".
- Used a combination of 3D FEM simulations (Shanghai supercomputer) and experimental methods.

How will we reproduce the results?

- Briefly derive the Topographic wave equation (TWE) and solve using spectral numerical methods.
- Run eigenvalue analysis and plot streamline, vorticity and energy results.

Why is this set up important?

- Non-axisymmetric structure of the geometries means the classical linear theory developed by Howard and Greenspan [2] cannot be applied.
- This motivates the search for alternative methods to solve the spin-up/down problem.

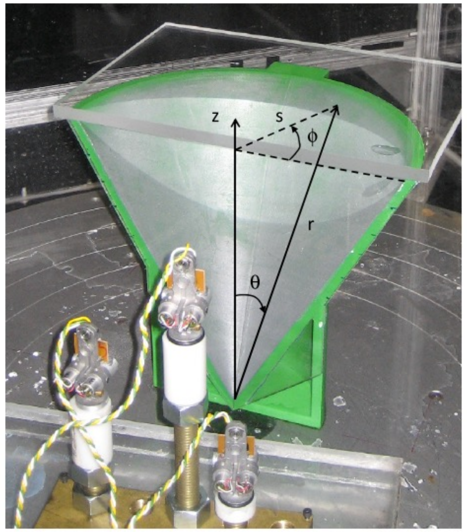


Fig 1. Experimental setup in [1].

What is spin-down?

- A spin-down problem is concerned with understanding the transient dynamics of a fluid following a fixed decrease in its rotation rate from $\boldsymbol{\Omega_0}$ to $\boldsymbol{\Omega_1} = (1 - \varepsilon)\boldsymbol{\Omega_0}$ for some small $0 < \epsilon \ll 1$.
- The Rossby number is a measure of the scale of the change: $Ro = \frac{\Omega_1 - \Omega_0}{\Omega_1}$.

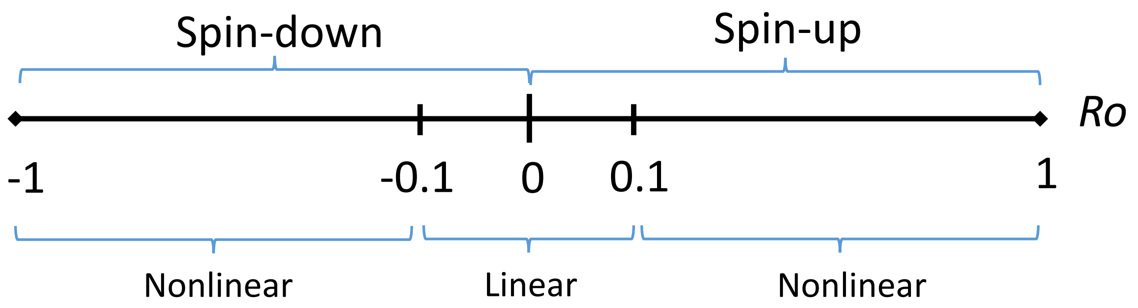


Fig 2. The Rossby number scale for spin-up and spin-down.

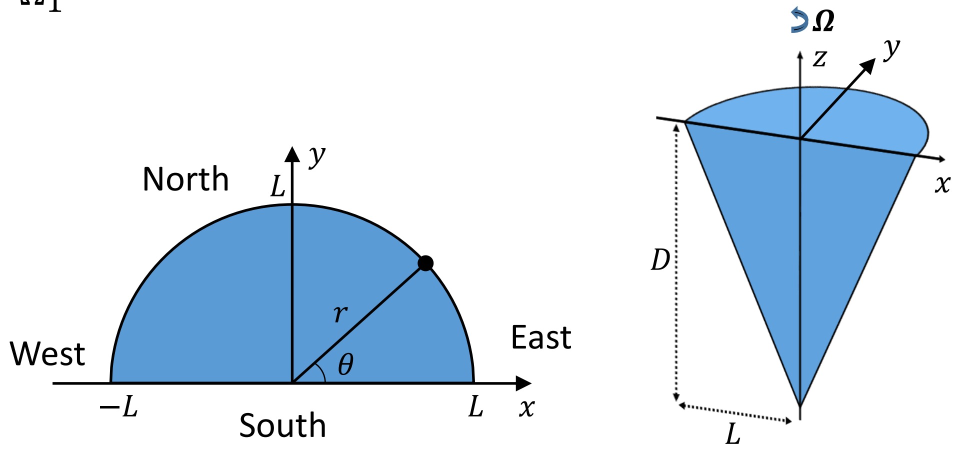


Fig 3. The 2D and 3D diagrams of the half cone.

- It is important as the process arises in astrophysical, geophysical and industrial systems.

Inviscid Half Cone - Problem Derivation

Navier-Stokes Equations

$$\frac{D\boldsymbol{u}}{Dt} + (\Omega_1 \hat{\boldsymbol{z}} \times \boldsymbol{u}) = -\frac{1}{\rho}\nabla p + \upsilon \nabla^2 \boldsymbol{u}$$

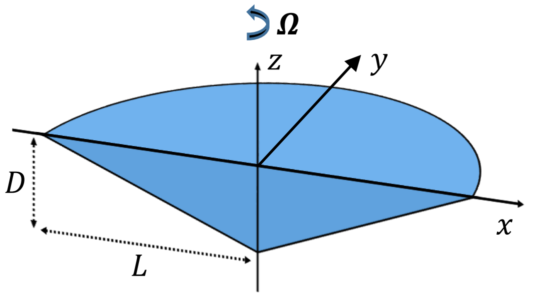
$$\nabla \cdot \boldsymbol{u} = 0$$

Shallow water assumption

$$\delta = \frac{D}{L} \ll 1$$

(Ignore nonlinear effects)

(Omit viscosity for now)



Linearised Shallow Water Equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial H}{\partial t} + \nabla \cdot (\boldsymbol{u}H_{\boldsymbol{0}}) = 0$$

Rigid lid $\Rightarrow \; = 0 \qquad \Rightarrow \; \boldsymbol{u} = \frac{1}{H_0}\begin{pmatrix} -\psi_y \\ \psi_x \end{pmatrix}$

Depth Profile $(H_0(x, y))$

Conservation of potential vorticity

$$\frac{D}{Dt}\left(\frac{\zeta + f}{H_0}\right) = 0$$

(Ignore nonlinear effects)

(Scale length/depth/Coriolis: $\mathrm{L} = D = f = 1$)

(Inviscid) Topographic Wave Equation

$$\frac{\partial}{\partial t}\nabla \cdot \left(\frac{\nabla\psi}{H_0}\right) + \hat{\boldsymbol{z}} \cdot \left(\nabla\psi \times \nabla\frac{1}{H_0}\right) = 0 \;\; in \;\; V$$

$$\psi = 0 \;\; on \;\; \partial V$$

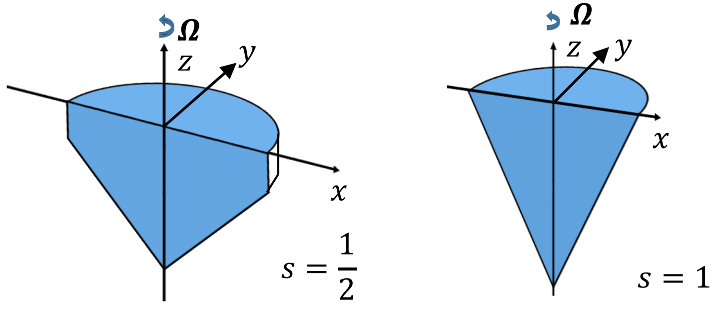
Harmonic Solution

$$\psi = \phi(r, \theta)e^{i\omega t}$$

Cylindrical Transform
(Eigenvalue Problem)

$$irH'_0\phi_\theta = \omega[r^2 H_0\phi_{rr} + (rH_0 - r^2H_0')\phi_r + H_0\phi_{\theta\theta}]$$

Depth Profile: $H_0(r) = 1 - sr$



Inviscid Half Cone - Spectral Method

- Very similar to the finite element method except the solution is expressed as sum of non-zero basis functions over entire domain (instead of just the finite elements) [3].
- Allows us to discretise the domain and differentiate using *Chebyshev differentiation matrices* D_θ *and* D_r.

Discretise using M+1 and N+1 Chebyshev points in each direction:

$$\theta_i = \cos\left(\frac{i\pi}{M}\right) \text{ for } i = 0, \dots, M$$

$$r_j = \cos\left(\frac{j\pi}{N}\right) \text{ for } j = 0, \dots, \mathrm{N}$$

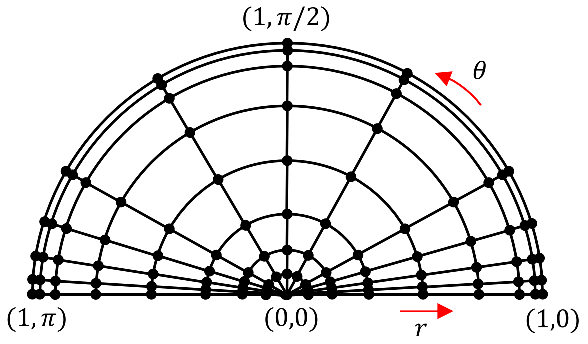


Fig 4. The Chebyshev mesh.

Chebyshev differentiation matrix definition

$$D_\theta = \begin{cases} \dfrac{(2M^2+1)}{6} & for \;\; i = j = 0 \\ \dfrac{-(2M^2+1)}{6} & for \;\; i = j = M \\ \dfrac{-\theta_i}{2(1-\theta_i^2)} & for \;\; i = j = 1, \dots, M-1 \\ \dfrac{c_i(-1)^{i+j}}{c_j(\theta_i - \theta_j)} & for \;\; i \neq j = 0, \dots, M \end{cases}$$

$$c_i = \begin{cases} 2 & i = 1, M \\ 1 & otherwise \end{cases}$$

Kronecker Product (Differentiate in 2D)

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,q}B \\ \vdots & \ddots & \vdots \\ a_{p,1}B & \cdots & a_{p,q}B \end{bmatrix}$$

$$\frac{\partial^n}{\partial\theta^n} \rightarrow I_{N+1} \otimes D_\theta^n$$

$$\frac{\partial^n}{\partial r^n} \rightarrow D_r^n \otimes I_{M+1}$$

Matrix Eigenvalue Problem

$$A\boldsymbol{\phi} = \omega B\boldsymbol{\phi}$$

Solve in MATLAB using 'eig.m'

Differential Eigenvalue Problem

$$irH'_0\phi_\theta = \omega[r^2H_0\phi_{rr} + (rH_0 - r^2H_0')\phi_r + H_0\phi_{\theta\theta}]$$

Inviscid Half Cone – Modal Solution

- Spurious eigenvalues arise as result of the discretisation and unexplained physics so we use methods from [5] and [6] to filter them out.
- Eigenvalues are computed at two different resolutions (M_1, N_1) and (M_2, N_2) and if they are very close in both solutions then they are deemed 'good', if not they are spurious.
- In [1], Li et al. claim that modes cannot exist because the vertex "precludes" their existence, this demonstrates this is not the case.

- Also introduce the initial vorticity condition: $\zeta(r,\theta,0) = \nabla \cdot \left(\frac{\nabla\psi}{H_0}\right) = \begin{cases} +1 \text{ for spin-down} \\ -1 \text{ for spin-up} \end{cases}$ at $t = 0$.

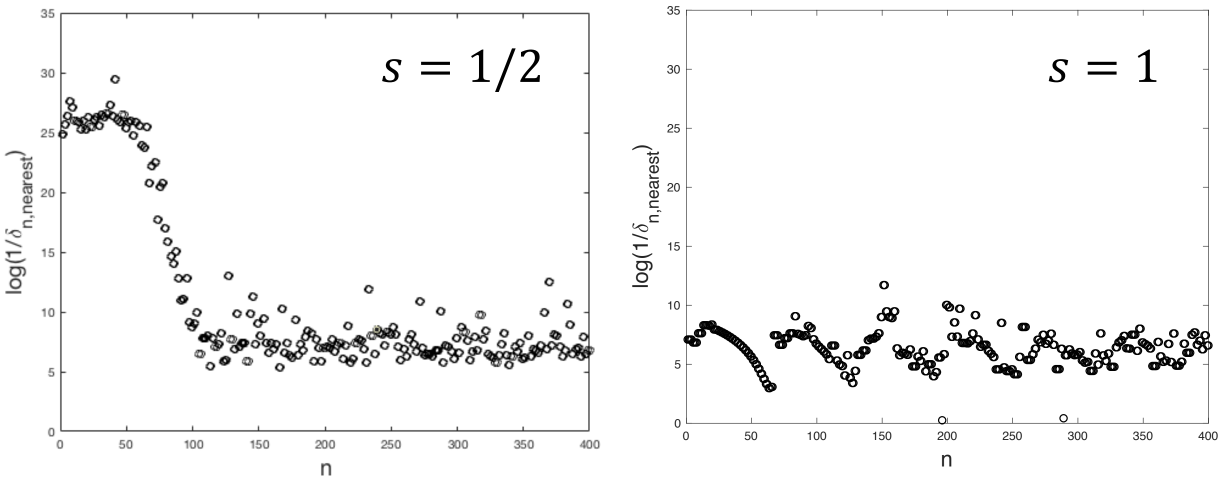


Fig 5. $\widehat{N} \approx 88$ 'good' modes exist (left). Zero 'good' modes (right).

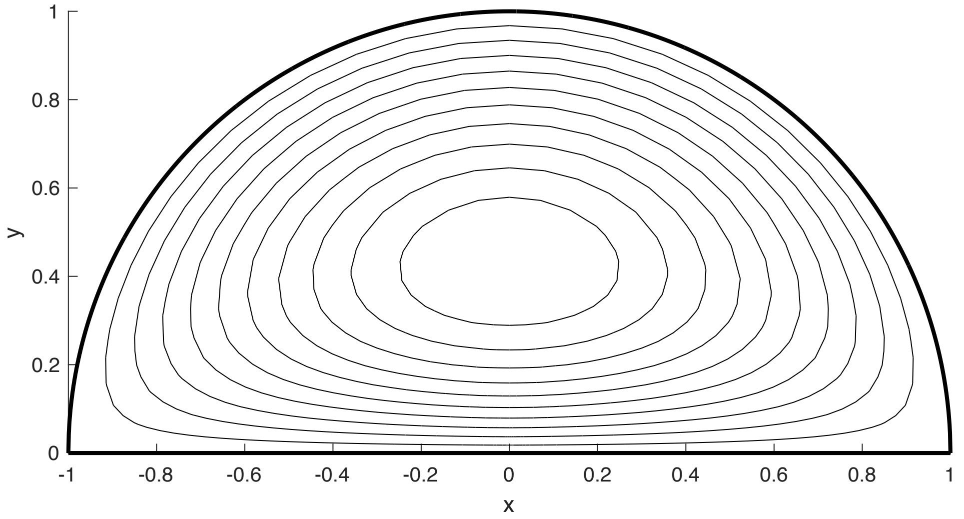


Fig 6. Streamlines generated by spin-down.

Inviscid Half Cone – Modal Solution

- Orthogonality analysis by Johnson [4] proves eigenvalues of inviscid TWE are *subinertial* ($|\omega| \leq f$) and real.
- Hence we can superpose 30 'good' modes ($\approx 75\%$ of total variance) resulting in the streamline patterns below.

$$\psi = \sum_{n=1}^{\hat{N}} a_n \phi_n(r,\theta) e^{i\omega_n t}$$

$$\zeta(r,\theta,0) = \nabla \cdot \left(\frac{\nabla\psi}{H_0}\right) = \begin{cases} +1 \text{ for spin-down} \\ -1 \text{ for spin-up} \end{cases} \text{at } t = 0$$

- The solution is not as accurate as we would like and does not fully capture the transient behavior of spin-down.

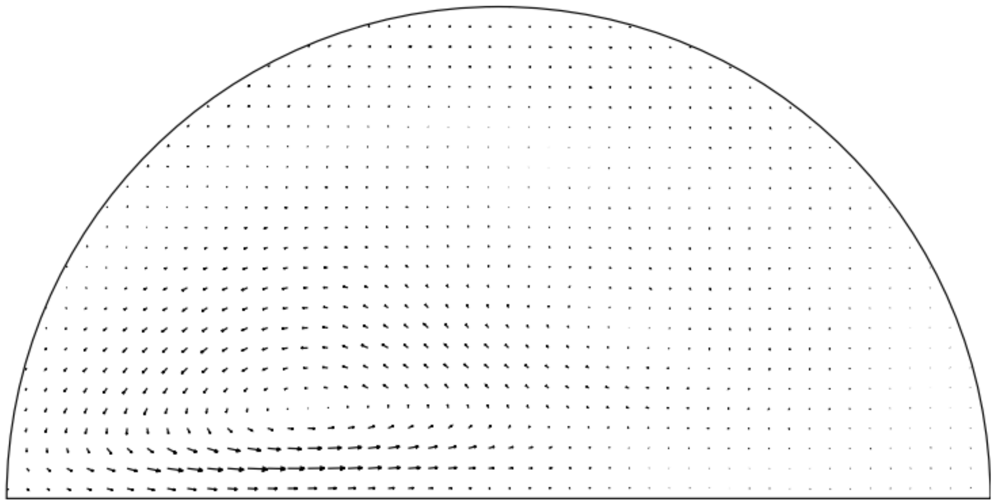


Fig 7. Horizontal velocity field, $\boldsymbol{u}(\boldsymbol{r},t)$, from the experiments in [1] for $Ro = -0.05$ at $t = 10$ (i.e. linear spin-down).

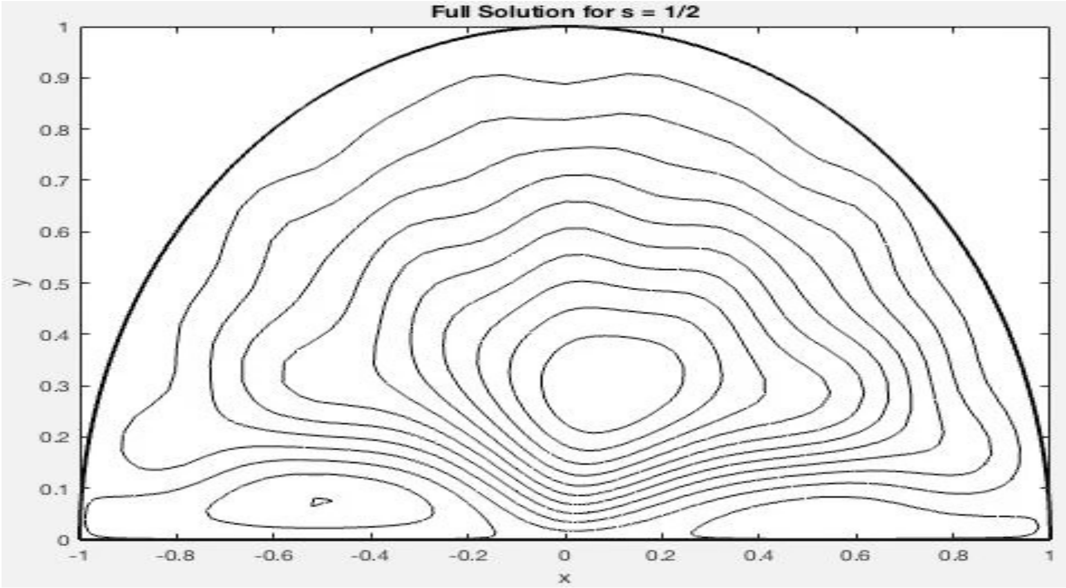


Fig 8. Superposed modal solution, $t \in [0,480]$.

Inviscid Half Cone – Time-dependent Solutions

- Instead of a harmonic solution we fully integrate over time, using $\psi = \psi(r, \theta, t)$ and the same spectral techniques as before plus an ODE solver in MATLAB.
- Much smoother streamlines, propagate westward where alternating cyclonic and anticyclonic cells build up with no way to dissipate (reflected in s = 1 case).
- Propagation occurs much faster in s = 1 than s = ½.
- Exactly the type of motion found in [1] except viscosity needs to be included.

Time-dependent Problem

$$\frac{\partial}{\partial t}[r^2 H_0 \partial_{rr} + (rH_0 - r^2 H_0')\partial_r + H_0 \partial_{\theta\theta}]\psi = -[rH_0' \partial_\theta]\psi \;\; in \;\; V$$

$$\psi = 0 \;\; on \;\; \partial V$$

$$\nabla \cdot \left(\frac{\nabla}{H_0}\right)\psi = 1 \;\; at \; t = 0^+$$

Matrix Problem

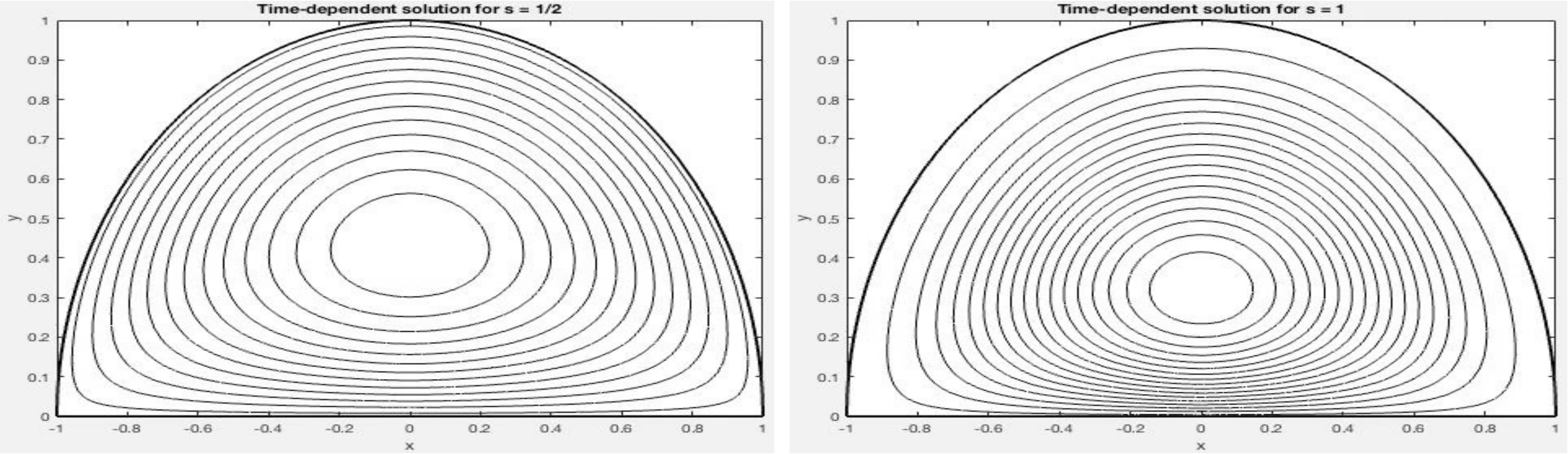
$$\frac{\partial}{\partial t} A\boldsymbol{\psi} = -B\boldsymbol{\psi} \;\; in \;\; V$$

$$\boldsymbol{\psi} = 0 \;\; on \;\; \partial V$$

$$C\boldsymbol{\psi} = \mathbf{1} \;\; at \; t = 0^+$$

Fig 9. Time-dependent integrated solution in $V = [0, \pi] \times [0,1]$, $t \in [0,700]$.

S = ½ (left)
S = 1 (right)



Viscous Half Cone

- Introduce a linear vorticity damping term scaled by the topography of the container $\Longrightarrow$ decay stronger in shallow areas ($\nu = 0.01$ for all following problems).
- As expected, no eigenvalues exist in the full slope case (s = 1), however they now have positive imaginary components in both cases $\Longrightarrow$ solutions decay because:

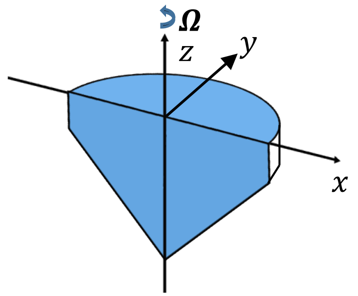
$$\psi = \phi(x,y)e^{i(a_n+ib_n)t} = \phi(x,y)e^{ia_nt}e^{-b_nt}.$$

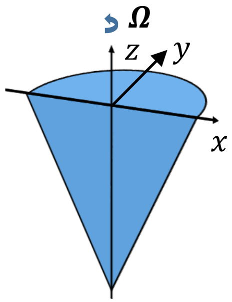
Mode (n)	$\omega_n\ (s=1/2)$	$\omega_n\ (s=1)$
1	$-0.08+0.01i$	$0.10+19.84i$
2	$0.08+0.01i$	$-0.10+19.84i$

- Even though spurious, eigenvalues in s = 1 case have much larger magnitude than the s = ½ $\Longrightarrow$ faster decay.
- Most likely because Ekman pumping on sloping bottom has more of an impact than the lateral walls.

(Viscous) Topographic Wave Equation

$$\frac{\partial}{\partial t}\nabla\cdot\left(\frac{\nabla\psi}{H_0}\right)+\hat{\mathbf{z}}\cdot\left(\nabla\psi\times\nabla\frac{1}{H_0}\right)=-\frac{\nu}{H_0}\nabla\cdot\left(\frac{\nabla\psi}{H_0}\right)\ in\ \ V$$
$$\psi = 0\ \ on\ \ \partial V$$





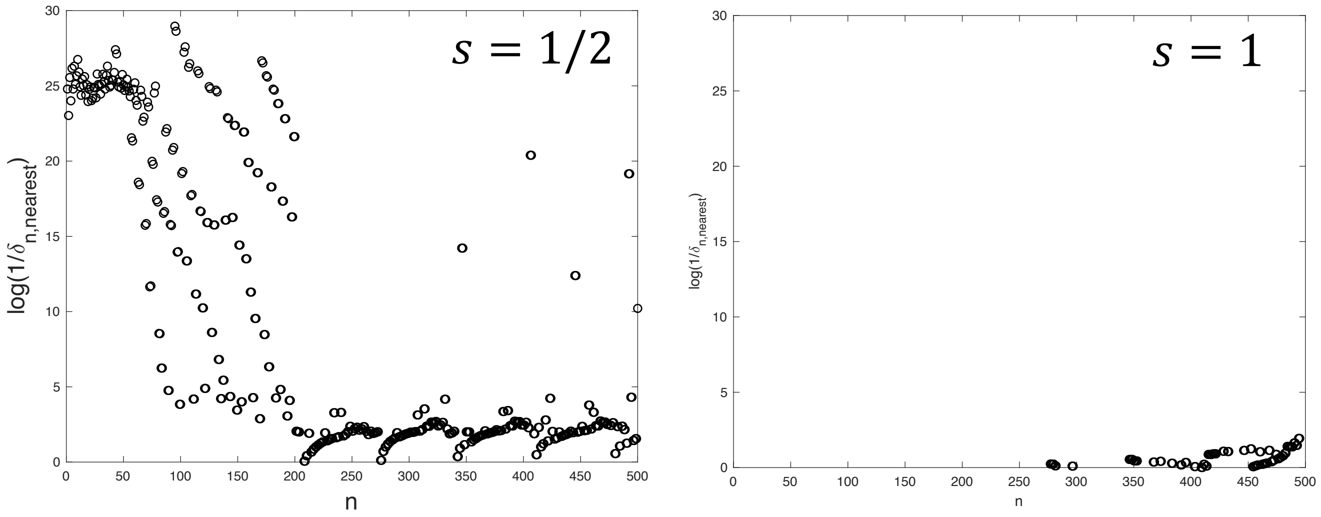


Fig 14. $\widehat{N}\approx 100$ ‘good’ modes exist (left). Zero ‘good’ modes (right).

Viscous Half Cone - Time-dependent Solutions

Time-dependent Problem

$$\frac{\partial}{\partial t}[r^2H_0^2\partial_{rr} + (rH_0^2 - r^2H_0'H_0)\partial_r + H_0^2\partial_{\theta\theta}]\psi = -[rH_0'H_0\partial_\theta + \nu(r^2H_0\partial_{rr} + (rH_0 - r^2H_0')\partial_r + H_0\partial_{\theta\theta})]\psi \ \ in \ \ V$$

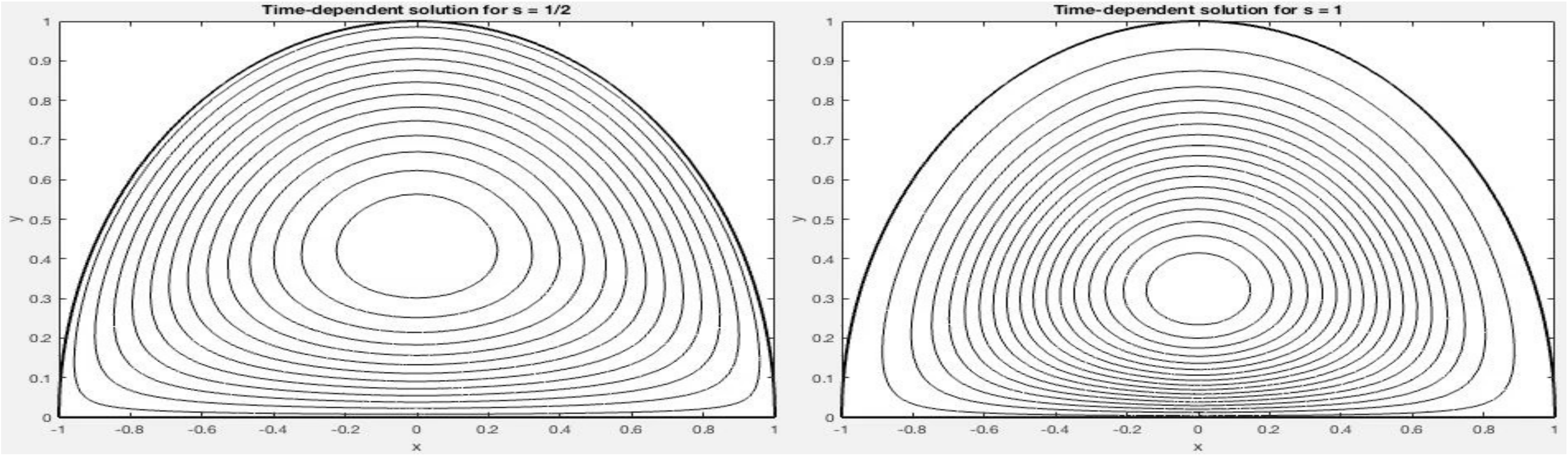
$$\psi = 0 \ \ on \ \ \partial V$$

$$\nabla \cdot \left(\frac{\nabla}{H_0}\right)\psi = 1 \ \ at \ t = 0^+$$

- As usual the waves propagate toward the south-western wall (much faster in the half cone) and the cells build up.
- The waves quickly dissipate as the Ekman layers rapidly spin-down the fluid, hence the waves no longer reflect back east.

Fig 15.
Time-dependent integrated solution $V = [0,\pi]\times[0,1]$, $t \in [0,700]$.

S = ½ (left)
S = 1 (right)



Viscous Half Cone – Vertical Vorticity

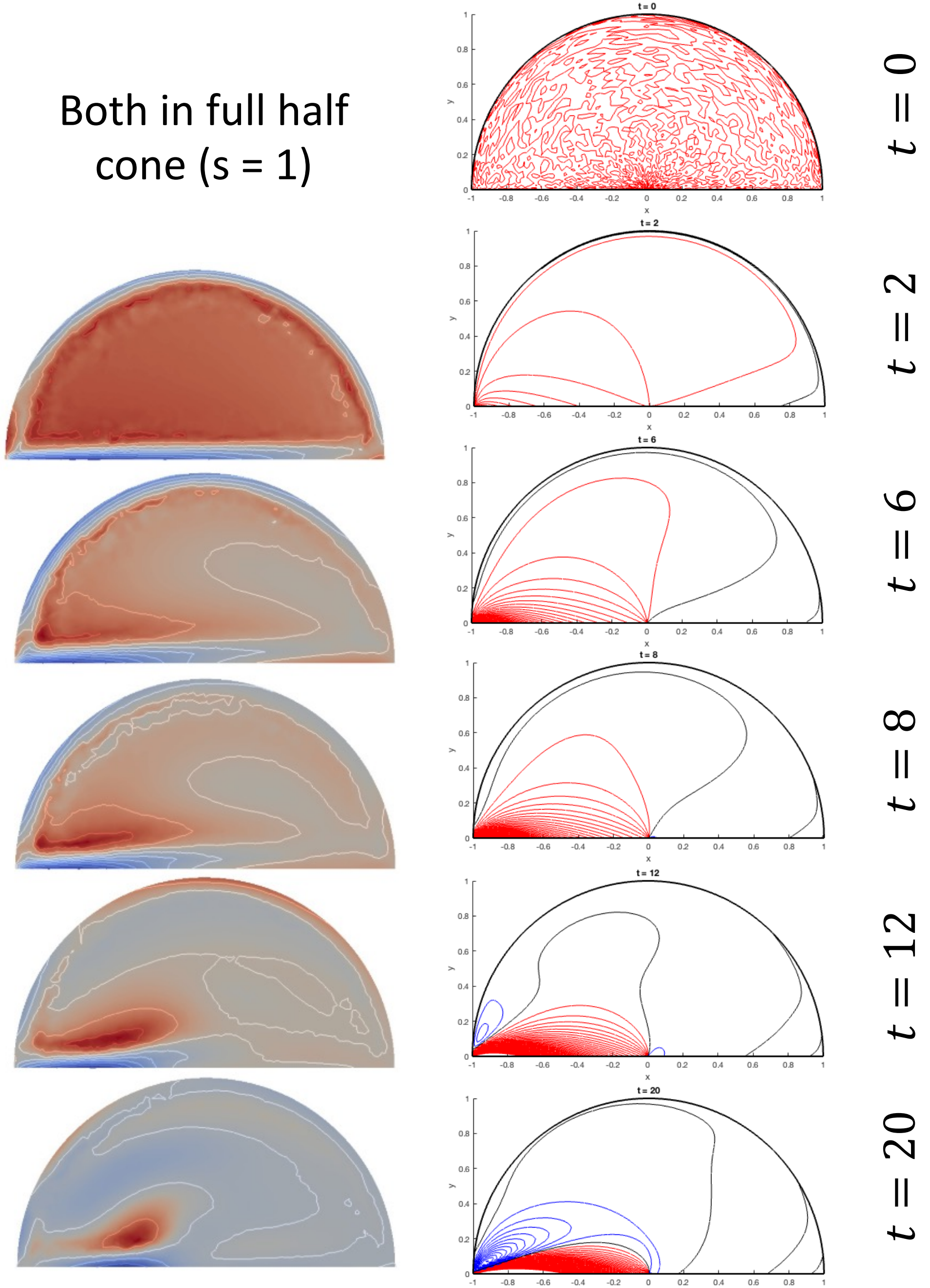
- From the streamfunction ψ, it is straightforward to calculate the *vertical vorticity* ζ at each time step t:

$$\zeta(r,\theta,t) = \nabla\cdot\left(\frac{\nabla\psi}{H_0}\right) = \frac{1}{r}\left(\frac{r}{H_0}\psi_r\right)_r + \frac{1}{r^2 H_0}\psi_{\theta\theta}$$

$$\rightarrow \quad \boldsymbol{\zeta}(r,\theta,t) = L\boldsymbol{\psi}(r,\theta,t)$$

- As expected, spin-down generates zero (almost negative) vorticity close to boundary and positive vorticity within interior at $t = 0$.
- For $t > 0$, initial vorticity propagates toward the south-western corner, surrounded by negative vorticity as it does.
- At the same time, excess vorticity is damped via Ekman pumping, returning the system to rigid body rotation.

Both in full half cone (s = 1)



(a) Results from [1].

(b) TWE results.

Fig 16. Contours of positive, negative and zero vorticity plotted in red, blue and black respectively. (a) Isolines from [1] at $z = -0.6$, for $Ro = -0.1$, values from [-21,9]. (b) Contours from the TWE, values from [-25,25].

Viscous Half Cone – Energy Decay

- In [1], Li et al. calculate the "speed of adjustment", or the time at which spin-up/down is concluded as the time taken t_N, for the kinetic energy of the system to drop to $1/N$th of its initial value. More formally:

$$\frac{\int_V |\boldsymbol{u}(t_N)|^2 \, dV}{\int_V |\boldsymbol{u}(0)|^2 \, dV} = \frac{1}{N}.$$

- They identify a linear relationship between t_N and the Ekman spin-up scale $E^{-1/2}$, more specifically, $t_{1000} \approx 0.1520E^{-1/2}$ and $t_{100} \approx 0.2386E^{-1/2}$.
- Replicating the analysis, linear relationships are found and, by averaging the ratios of the gradients, we find that spin-down occurs approximately 4.13 times faster in the half cone than the half cylinder with slope.

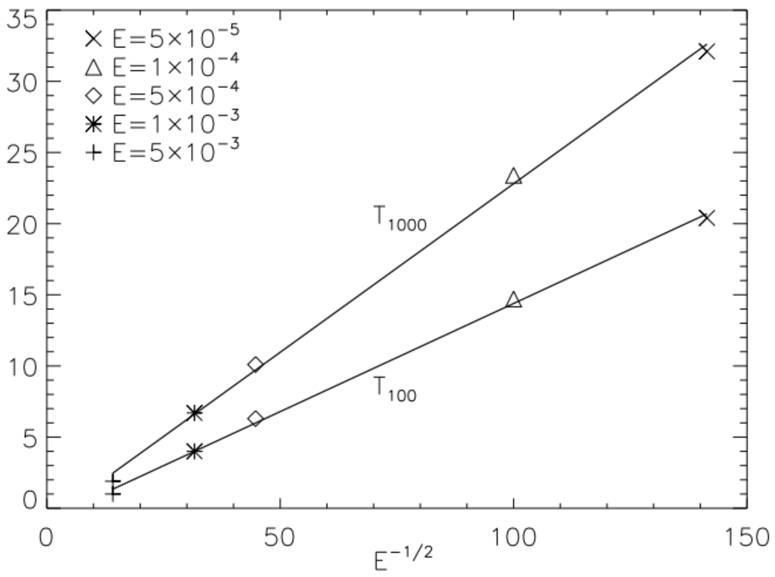


Fig 18. Energy decay times t_N against $E^{-1/2}$ from [1].

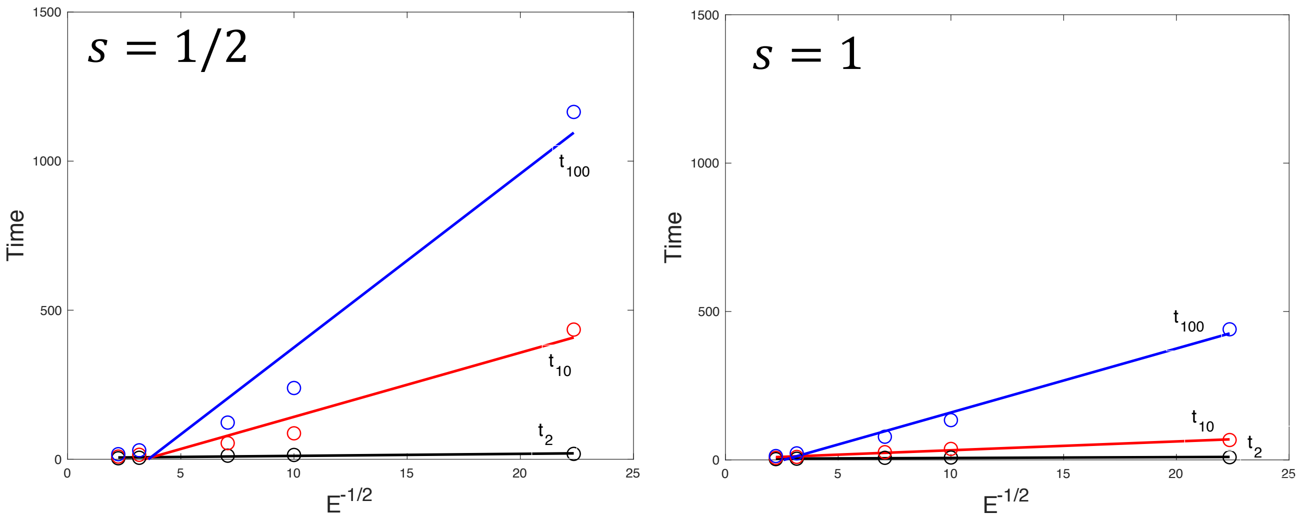


Fig 19. Energy decay times t_N against $E^{-1/2}$ in the half cylinder w/slope (s=1/2) and half cone (s=1).

Summary of Findings

Main Findings

- Accurately managed to reproduce the results of streamlines, vertical vorticity, vertical velocity and energy decay detailed in [1] using the two-dimensional topographic wave equation.
- Hence computational requirements were significantly reduced.
- It is determined that *no oscillatory modes exist* when the depth of the container goes to zero (as the energy integral becomes unbounded), countering the claim in [1] that the half cone vertex "precludes" their existence.
- Reflected waves only exist in the models where viscosity is absent (as short waves are too weak to be reflected).
- The linear spin-up analysis ($|Ro| \leq 0.1$) can be replicated and is identical to linear spin-down except that the sign of the streamfunction and hence velocity vectors and vorticity have opposite sign – as demonstrated in [1].

Further Work

- Study the effects of fluid stratification, compressibility and free surfaces.
- Can study other geometries (in the shallow water limit) for linear spin-up/down.
- Could look at 'spin-over' problems.

References

[1] L. Li, M. D. Patterson, K. Zhang, and R. R. Kerswell. Spin-up and spin-down in a half cone: A pathological situation or not? Physics of Fluids, 24(11):116601, 2012.

[2] H.P. Greenspan and L.N. Howard. On a time-dependent motion of a rotating fluid.

Journal of Fluid Mechanics, 17(3):385–404, 1963.

[3] L.N. Trefethen. Spectral Methods in MATLAB. SIAM, Philadelphia, 2000.

[4] E. R. Johnson. Topographic waves in open domains. Part 1. Boundary conditions and frequency estimates. Journal of Fluid Mechanics, 200(-1):69, 1989.

[5] J. P. Boyd. Chebyshev and Fourier Spectral Methods. Courier Corporation, second edition, 2001.

[6] E. R. Johnson and J. T. Rodney. Spectral methods for coastal-trapped waves. Continental Shelf Research, 31(14):1481–1489, 2011.

Viscous Half Cone – Vertical Velocity

- Similarly we can calculate the *vertical velocity* $w(r, \theta, z, t)$ at each time step t:

$$u_x + v_y + w_z = 0 \quad \Rightarrow \quad w_z = -\nabla \cdot \boldsymbol{u}$$

$$\int_z^0 w_{z'}\, dz' = -\int_z^0 \nabla \cdot \boldsymbol{u}\, dz' \quad \Rightarrow \quad w(z) = -\mathrm{z}\, \nabla \cdot \boldsymbol{u}$$

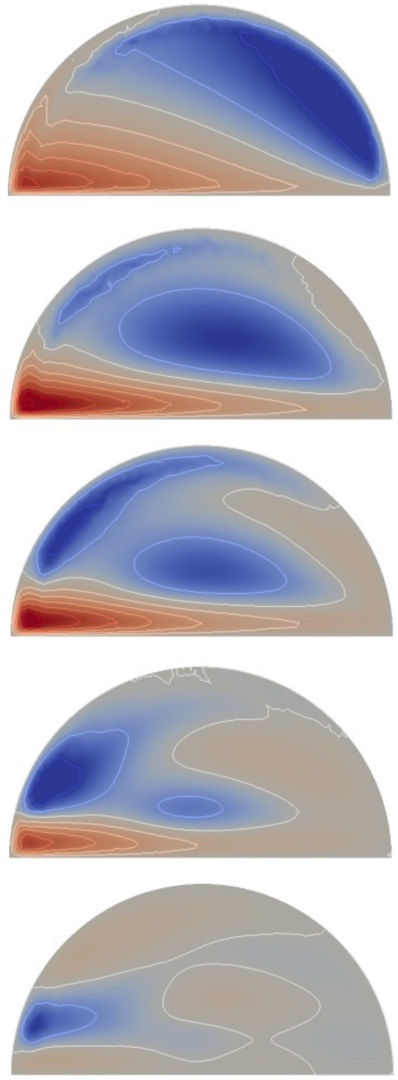
$$w(z) = -\frac{z}{H_0} H_0 \boldsymbol{\nabla} \cdot \boldsymbol{u} \quad \Rightarrow \quad w(z) = -\mathrm{z}\, \frac{H_0'}{rH_0^2} \psi_\theta$$

$$\rightarrow \quad \boldsymbol{w}(r, \theta, z, t) = L\boldsymbol{\psi}(r, \theta, t)$$

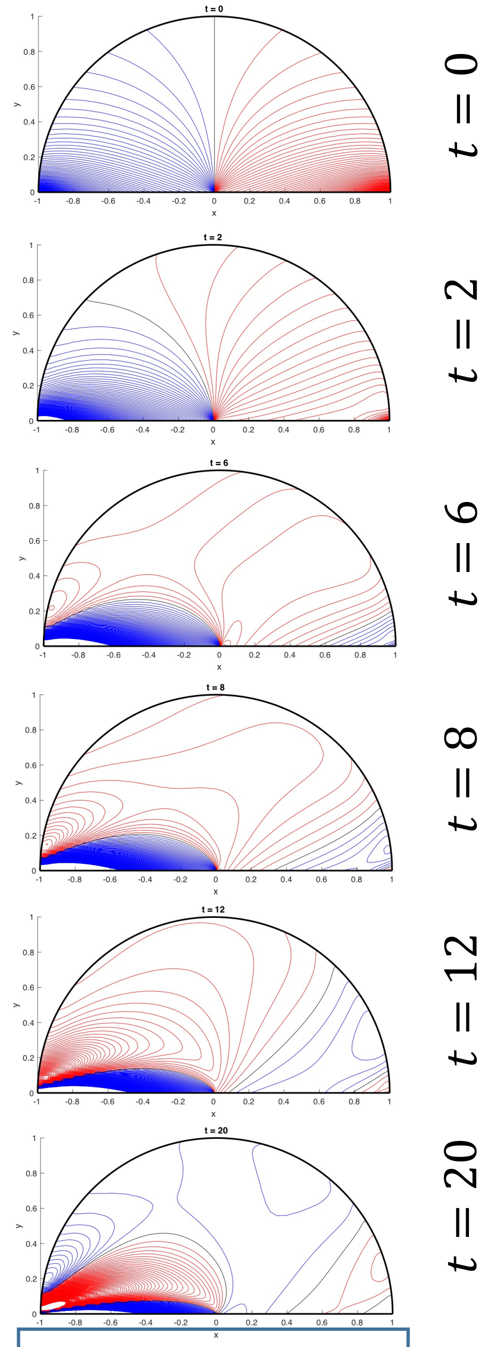
- In [1], it is shown that dynamics for linear spin-up/down ($|Ro| \leq 0.1$) are the same, hence we compare with spin-up computed in [1].
- Dynamics produced are identical, note however the colours must be reversed.

Fig 17. Contours of positive, negative and zero velocity plotted in red, blue and black respectively. (a) Isolines from [1] at $z = 0.6$, for $Ro = 0.01$, values from [-0.15,0.35]. (b) Contours from the TWE at $\mathrm{z} = -0.6$, values from [-1,1].

Both in full half cone (s = 1)



(a) Results from [1].



(b) TWE results.